

Motion of a thin spherically symmetric Shell of Dust in the Schwarzschild field

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Abstract

Summary. The equation of motion announced in the title was already deduced for the cases the inner metric being flat and the shell being negligibly small (test matter), using surface layers and geodesic trajectories resp. Here we derive the general equation of motion and solve it in closed form for the case of parabolic motion. Especially the motion near the horizon and near the singularity are examined.

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1

Motion represents a problem in General Relativity even in such cases where the particle's interaction is gravitational only. Because exact solutions of the problem are very rare many approximate solutions have been calculated, but they are often unsatisfactory for various reasons. On the other hand, surface layers are exact solutions of Einstein's field equations, if these equations are written as distribution equations. Furthermore, considering spherically symmetric configurations the problem of gravitational radiation does not arise.

Therefore it is appropriate to consider the general motion of a spherically symmetric surface layer.

2

Surface layers can be thought as strata of matter with a negligible small thickness; they can be represented by (non-spurious) jumps of the Christoffel affinities – their derivatives entering the energy-momentum tensor yield δ -distributions. The related formulas are deduced in [1], [2] and [3]. Another but equivalent approach [4] uses the second fundamental tensor and will be applied here to deduce the above mentioned motion. The cases where the interior metric is flat and the matter is dust are treated in [4]. In [5], additionally, charges have been considered. A static configuration is discussed in [7], and the tangential pressure is due to collisions of particles moving on circular orbits there.

3

Now let Σ be the spherically symmetric time-like hypersurface at which matter is concentrated. Inside Σ we take proper time T and usual angles ψ, φ as coordinates, $d\Omega^2 = d\psi^2 + \sin^2 \psi d\varphi^2$. The invariant surface of the sphere $T = \text{const.}$ will be denoted by $4\pi R^2$, $R = R(T)$. Then the first fundamental tensor of Σ is $g_{\alpha\beta}$, and

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = -dT^2 + R^2 d\Omega^2, \quad \xi^0 = T, \quad \xi^2 = \psi, \quad \xi^3 = \varphi.$$

Outside Σ one has to use the Schwarzschild metric with mass parameters M_- and M_+ in $V_- = [r < R]$ and $V_+ = [r > R]$ resp. and to match them together in a continuous manner.

Energy conditions require $M_+ \geq M_- \geq 0$, M_- is the black hole's mass and $M_+ - M_-$ is the shell's gravitational mass.

The energy momentum tensor is $(\xi^1 = r)$, $(i, k = 0, 1, 2, 3)$

$$T_i^k = \delta(r - R) [\text{diag}(-\mu, 0, p, p)]_i^k$$

with $(G = c = 1)$

$$\mu = \frac{W_- - W_+}{4\pi R}, \quad W_\pm = \sqrt{1 + (dR/dT)^2 - 2M_\pm/R}$$

and

$$8\pi p = \left[d^2 R / dT^2 + (1 + (dR/dT)^2) / R \right] (W_+^{-1} - W_-^{-1}) + \frac{M_-}{R^2 W_-} - \frac{M_+}{R^2 W_+}$$

(cf. [8] for details). There is no T_1^k -component because by construction no matter flow orthogonal to Σ exists, and the tangential pressure is isotropic because of spherical symmetry.

Requiring an equation of state $p = \alpha\mu$ one obtains a differential equation for the function $R(T)$:

$$R d^2 R / dT^2 = 2\alpha W_+ W_- - \frac{M_+ W_- + M_- W_+}{R(W_+ + W_-)}.$$

The conservation law $T_{0;i}^i = 0$ leads to

$$\mu \cdot 4\pi R^2 R^{2\alpha} = E = \text{const.},$$

and therefore

$$R(dR/dT)^2 = M_+ + M_- + \frac{E^2}{4R^{4\alpha+1}} + (M_+ - M_-)^2 R^{4\alpha+1} / E^2 - R.$$

For $\alpha = 0$, there of course $E = M_+ - M_-$, we have the case of matter being dust. For $p \leq 0$, the shell will collapse into the singularity after a finite proper time interval.

Parabolic motion takes place if $dR/dT \rightarrow 0$ as $R \rightarrow \infty$ and is therefore possible for $\alpha = 0$ only. For this case the solution, which describes the fall into $R = 0$ at $T = 0$, is

$$R(T) = \frac{H \cdot \sqrt[3]{(M_+ + M_-)/2}}{\sqrt[3]{-3T - K + H} - \sqrt[3]{-3T - K - H}}$$

where

$$H = \sqrt{9T^2 + 6KT} \quad \text{and} \quad 2K = (M_+ - M_-)^3 / (M_+ + M_-)^2.$$

4

The horizon is not a singular surface (but its geometry has an interesting shape, cf. [8]) and therefore its crossing cannot be measured locally. This statement keeps valid if *locally* means *all information from a neighbourhood*, e.g. all curvature.

5

The limiting process from a gravitational field creating particle to a test particle leads to geodesic motion (cf. e.g. [6]), but in the proof there is presumed the curvature to be bounded in a neighbourhood of the particle's world line. The question arises whether this is an essential presumption or not. Here we have a similar configuration: By a formal inserting of $\alpha = E = 0$ one obtains the equation

$$Rd^2R/dT^2 = -M_+/R,$$

which is indeed the equation of a radially falling geodesic test particle. But in the limiting process $E \rightarrow 0$, $R = 0$ is a singular point of the differential equation, and indeed, the leading terms in a neighbourhood of $T = 0$ are

$$R(T) \sim T^{1/2} \quad \text{for each} \quad M_+ > M_-$$

and

$$R(T) \sim T^{2/3} \quad \text{for} \quad M_+ = M_-.$$

The typical radius where the test particle's motion and the layer's motion begin to differ is

$$R(T) \approx \frac{(M_+ - M_-)^2}{M_+}.$$

This means the dust strata approach the singularity with another power of proper time than the related geodesic test particle does. This supports the common opinion on this question by a sequence of exact solutions.

References

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Note added in 2014: Reference [8] appeared as H.-J. Schmidt: Surface layers in General Relativity and their relation to surface tensions, Gen. Rel. Grav. **16** (1984) 1053 - 1061; see arXiv:gr-qc/0105106v1 for a reprint.